Introduction to Sigma Notation

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What is sigma notation?

Σ is the capital Greek letter for the sound 's'. In this case, it's just shorthand for "sum". Sigma notation is what we use when we have a series of numbers that need to be summed. It's that simple.

This is a really common scenario in data science and statistics, that's why the notation exists. You may not want to be deriving equations yourself, but it's really helpful to know what someone else is talking about.

There's a few quirks and rules of notation that are commonly used that are helpful to know. Here they are!

What are the bits?

Sigma notation isn't just about the Σ. It comes with accoutrements that describe where the series to sum starts, where it ends. It also describes whether you want a plain-vanilla 'add the numbers up', or if you're going to add toppings - multiply the numbers by something, square them, take something away etc.

The basic format of sigma notation is this:

$$\sum_{i=1}^{n} x_i$$

There are alot of parts to this one, let's define them one by one:

- Σ is simply the operator. It's not a variable or anything you change, it's a symbol like + or -. It's short hand for a whole lot of + signs actually!
- x is the series that will be summed. There are a whole bunch of xs and they have an order. To keep that order straight, they have subscripts (the little number next to them).
- i is the subscript of x, it's a convention that gives each member of the series x a name, in this case $i = 1, 2, 3, \ldots n$. Here, n is the last member of the series. It could be 10, 20, 100 or even infinity. The … between 3 and n just stands for all the numbers between them that are missing.

The piece of sigma notation we have above can be expressed as a simple sum. I highly recommend that when you are trying to understand something, you simply write out the sum to begin with. Here's what that would look like here:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + x_4 + \ldots + x_n$$

The $\ldots$ indicates there are members of this series we haven't written out explicitly - in this case we don't really know what n is so we can't write out the whole series.

Let's take some concrete examples to make this more clear.

1. If $n = 5$ let's write out $\sum_{i=1}^{n} x_i$ in long form.
    - $\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + x_4 + x_5$
2. If $m = 2$ let's write out $\sum_{j=1}^{m} y_i$ in long form.
    - See what happened here? We gave the variable a different name, y, the subscript y a different name, j, and the end point of the series, m, a new name too. It all works in precisely the same way:

$$\sum_{j=1}^{m} y_i = y_1 + y_2$$

Let's put some numbers in to play

We now understand what all the 'bits' are. Let's see what happens with real numbers.

Let's say that $x = 2, 4, 5, 6$. All this is saying is that x is a series of numbers. In this case $x_1 = 2$, $x_2 = 4$, $x_3 = 5$ and $x_4 = 6$. If we wanted to add them all together, we could say that $\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + x_4$ where $n = 4$. This could be written out as:

$$\begin{aligned}\sum_{i=1}^{n} x_i &= x_1 + x_2 + x_3 + x_4 \\ &= 2 + 4 + 5 + 6 \\ &= 17\end{aligned}$$

Let's try another example. Say that $y = 1, 1, 0, 1, 0$. In this case $y_1 = 1, y_2 = 1, y_3 = 0, y_4 = 1, y_5 = 0$. If we wanted to add them all together, we could say that $\sum_{i=1}^{n} y_i = y_1 + y_2 + y_3 + y_4 + y_5$ where $n = 5$. This could be written out as:

$$\begin{aligned}\sum_{i=1}^{n} y_i &= y_1 + y_2 + y_3 + y_4 + y_5 \\ &= 1 + 1 + 0 + 1 + 0 \\ &= 3\end{aligned}$$

Constants can be treated specially in sigma notation

Constants are just a number that does not change, it is not part of a series: it just lives on its own.

Think about this: $\sum_{i=1}^{n}(x_i + 2)$. This is the same thing as saying

$$\sum_{i=1}^{n}(x_i + 2)$$
$$= (x_1 + 2) + (x_2 + 2) + (x_3 + 2) + \ldots + (x_n + 2)$$

We have added 2 to this summation n times. So we could simply rewrite this as

$$\sum_{i=1}^{n}(x_i + 2) = 2n + \sum_{i=1}^{n} x_i$$

.

In general, if a is a constant, then $\sum_{i=1}^{n}(x_i + a) = \sum_{i=1}^{n} x_i + an$

Let's consider another scenario where constants are inside a summation of a series. If we have $\sum_{i=1}^{n}(2x_i)$, we could rewrite this as

$$\sum_{i=1}^{n}(2x_i)$$
$$= 2x_1 + 2x_2 + 2x_3 + \ldots + 2x_n$$

We just multiplied each term by 2. It would mean exactly the same thing to pull this to the outside of the summation like this

$$\sum_{i=1}^{n}(2x_i)$$
$$= 2\sum_{i=1}^{n}(x_i)$$

.

In general, if a is a constant, then

$$\sum_{i=1}^{n}(ax_i)$$
$$= a\sum_{i=1}^{n}(x_i)$$

We need some examples!

Say that $y = 1, 1, 0, 1, 0$. In this case, what is $\sum_{i=1}^{n}(3x_i)$?

In general, when an end point is not specified, we assume we are to add the whole series. So in this case, $n = 5$.

$$\sum_{i=1}^{n}(3x_i)$$
$$= 3\sum_{i=1}^{n}(x_i)$$
$$= 3(1+1+0+1+0)$$
$$= 3(3)$$
$$= 9$$

Let's try another.

If $x = 2, 4, 5, 6$, let's find $\sum_{i=1}^{n}(\frac{1}{n}x_i)$. This is a little different, but the same principles apply. Here we will take $n = 4$. Since $\frac{1}{4}$ is a constant, it can be brought to the front of the summation like this:

$$\sum_{i=1}^{n}(\frac{1}{n}x_i)$$
$$= \frac{1}{n}\sum_{i=1}^{n}x_i$$
$$= \frac{1}{4}(2+4+5+6)$$
$$= \frac{1}{4}(17)$$
$$= 4.25$$

Congratulations, you just calculated your first average with sigma notation!

We can use sigma notation on more than one series at the same time

What if you have two series that interact with each other? Summation notation is used alot in this context. You may see something like:

$$\sum_{i=1}^{n}x_iy_i$$

This simply means that we multiply each matching term of x and y for all $i = 1, 2, \ldots, n$. The equation then looks like this:

$$\sum_{i=1}^{n}x_iy_i$$
$$= x_1y_1 + x_2y_2 + x_3y_3 + \ldots + x_ny_n$$

Let's take a concrete example. If $m = 4$, $x = 2, 4, 5, 6$ and $y = 1, 1, 0, 1, 0$ then we could find $\sum_{i=1}^{m}x_iy_i$.

Note how $m = 4$ here. We couldn't use $n = 5$ because the series x only has four numbers in it!

Let's work this one out:

$$\sum_{i=1}^{m} x_i y_i$$
$$= 2(1) + 4(1) + 0(5) + 6(1)$$
$$= 2 + 4 + 0 + 6$$
$$= 12$$

There are lots of uses for sigma notation

Sigma notation is used all over statistics and data science in different forms. You'll see terms squared, added, divided by and so on. If you keep these basic rules in mind, you'll have a more clear idea of what's going on. If you'd like to see some more information on sigma notation, try this site from Columbia (http://www.columbia.edu/itc/sipa/math/summation.html).