An Introduction to Algebra

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Welcome to the world of mathematics

So it turns out, despite your mathematics teacher's best intentions, you didn't go on to use maths in your every day life. Maybe you hated it as a school kid and developed a reflexive belief over time you weren't good at it.

Only, now your life has moved on and you need some maths. Maybe you're taking a course at university and it's literally all in Greek. Maybe your kid's maths homework is getting scary. Maybe you just want to know if you can.

This isn't a very formal introduction to the basic algebra we use as data scientists, because if you need this appendix, you possibly already had a formal introduction and hated it, didn't understand it, ignored it or some combination of the three. Also, I'm not a formal person, so there's that.

I've taught matehmatics to developing econometricians and statisticians for longer than I care to recall. I started out early and was a truly terrible teacher. It turns out that the way I learned maths wasn't normal at all.

The intervening fifteen years has been a productive data collection period and I'm much better at it now.

I spend alot of time helping people who are prepared to work hard to succeed and just need some kind of resource to fill in the gaps. So that's the plan here- gap filling. Take what's useful to you and skip over the rest. I'll start from the simpler concepts and build up.

Douglas Adams of *Hitchiker's Guide to the Galaxy* fame had it right in *Dirk Gently's Holistic Detective Agency*. Discussing the mathematical complexity of the natural world, he wrote:

> … the mind is capable of understanding these matters in all their complexity and in all their simplicity. A ball flying through the air is responding to the force and direction with which it was thrown, the action of gravity, the friction of the air which it must expend its energy on overcoming, the turbulence of the air around its surface, and the rate and direction of the ball's spin. And yet, someone who might have difficulty consciously trying to work out what 3 x 4 x 5 comes to would have no trouble in doing differential calculus and a whole host of related calculations so astoundingly fast that they can actually catch a flying ball.

If you can catch a ball, you are performing complex calculus instinctively. All we are doing in formal mathematics and data science is putting symbols and a syntax around the same processes you use to catch that ball.

Maybe you've spent a lot of your life believing you "can't" or are "not good at" mathematics, statistics or whatever part of the computational arts is getting to you. These are concepts we begin to internalise at a very early age and often carry them through our lives.

The good news is *yes you can*. If you can catch that ball (occasionally at least. Or in my case, sometimes I'm close!) then there is a way for you to learn mathematics, data science and all the things that go with it. It's just a matter of finding the one that works for you.

Yes you can.

Let's start from the very beginning

Maria Von Trapp had it right - let's start from the very beginning. Remember that part of the *Sound of Music* where she has a whole bunch of kids learning music in a rousing four minute sing-a-long number? Well, there's a lot of maths in music. We could easily have changed out *do re me* for *1 2 3*. Let's start at the beginning, and we won't dance.

The concept of algebra

The whole purpose of algebra is to generalise mathematics from numbers we do know about - the ones we can easily see and work work with - to those numbers we don't have much information on or can't observe. Algebra gives us a set of tools to find out about these.

You can think about learning the basics of algebra like you do a language. If I dumped you in Paris with only the word *baguette* in your French vocabulary, you would have a pretty miserable, though probably quite glutenous, afternoon. On the other hand, six weeks later you'd have developed a more well rounded vocabulary, a good idea of the syntax and will have moved on from your diet of bread and river water.

So let's start with the simplest concepts in our vocabulary and build the language around them. This is just a refresher of a very wide-ranging and diverse field. You won't be a linear algebra expert after this, but you will start to be able to decode what's happening on the page when you see it.

Introducing x

Take x. You can think of x as just a place holder. It's just there until we can replace it with something better.

If we want to add or subtract from x we can do it in the normal way: $x - 5$ and $x + 2$. Once we can remove our placeholder x and replace with a number, it all works in the regular mathematical way.

Let's try some examples:

1. If $x = 2$, what is $x + 3$? Well, if $x = 2$, then

$$\begin{aligned} &x + 3 \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

2. If $x = 0.4$, what is $x - 1$? If $x = 0.4$, then

$$x - 1$$
$$= 0.4 - 1$$
$$= 0.6$$

Pro tip: Negative numbers don't come naturally yet? It's OK to use a calculator!

3. If $x = 7$ what is $x - 4$? If $x = 7$, then

$$x - 4$$
$$= 7 - 4$$
$$= 3$$

If x is just a placeholder, then really it could be any symbol at all. To make sure things stay clear, we tend to use letters rather than emoji, for example, but the basic concept remains the same.

We could do the same things with letters like a and b. There's nothing special about any of them at all. For example: 1. If $a = 10$ and $b = 20$, what is $a + b$? In this case, we can simply say that:

$$a + b$$
$$= 10 + 20$$
$$= 30$$

We can actually assign this outcome of $a + b$ to another letter, to use it in more complex equations. That is, we could say that $c = a + b = 30$. 1. If $x = 0.7$ and $y = 0.9$, what is $z = y - x$?

$$y - x$$
$$= 0.9 - 0.7$$
$$= 0.2$$

Multiplication and division

We can now do basic addition and subtraction in algebra. This is a great start! However, we typically need to be able to do multiplication and division as well.

We could write $xx2$ to mean "x times 2", but that would be unnecesarily confusing. So when combining multiplied terms in algebra, we just omit the operator (the "times" sign) and write $2x$. Typically, we put numbers in front of the letters.

Let's try a couple of examples: 1. If $x = 100$, what is $2x$?

$$2x = 2 * 100 = 200$$

1. If $a = 0.4$, what is $5a$? $5a = 5 * 0.4 = 3.2$

Division is often represented just by creating a fraction and placing the divisor (the thing you're dividing by) underneath like this: $\frac{2}{3} \approx 0.66666$.

Pro tip: $\approx$ just means "approximately equal to".

In algebra, we just put our quantity we are dividing on the top and the amount we're dividing by on the bottom. So "x divided by 2" is just $\frac{x}{2}$. Let's try a couple of examples: 1. If $x = 10$, what is $\frac{x}{2}$? In this case we have

$$\frac{x}{2}$$
$$= \frac{10}{2}$$
$$= 5$$

1. If $x = 5$, what is $\frac{x}{5}$? In this case we are looking at $\frac{5}{5} = 1$.

Exponentiation

This is just a complicated way of saying "to the power of". When we say that $2^2 = 4$ we are saying "two to the power of two = four". You can work out what 3^2 is on your calculator. But did you know that $\sqrt{9} = 9^{\frac{1}{2}} = 3$? Square roots are just exponentiation too.

Definition: Like Terms

A 'like term' is when two or more parts of your equation can be collected together. That means they refer to the same variable, perhaps x and will be of the same magnitude of that variable. E.g. all the terms that have x^2 or all the terms that have $y^{\frac{1}{2}}$.

The main purpose of talking about like terms, is because they are an excellent way to make an equation simpler. Anything that makes algebra simpler is something I am 100% in favour of, so let's take a look.

Take the equation

$$z = a + bx + cx^2 + 2x + 3 + 4x^2$$

This is rather a hot mess and it's really hard to see what's going on here. We need to simplify it.

Let's start by thinking about what kinds of terms are in this equation.

- We have z - it's just by itself with no other *variables* like it. We can't do anything to simplify it, so we'll leave it be.
- We have a, no others like it? We'll leave it be.
- We have 3. Well you know what we're going to do with that, don't you?
- We have bx and $2x$ can we simplify these? Yes we can. We can collect the terms together by taking the common factor, x out. So $bx + 2x = x(b + 2)$. We can put the x outside a pair of brackets and the other parts inside it. This means that x is multiplied by every term in the bracket. We took two terms and made one compound term out of them. This is a really useful thing in algebra!
- We also have cx^2 and $4x^2$. These two are also like terms. They are the same *order* in x, that is squared. We call these *quadratic in x*. We can do the same thing. We can take x^2 outside of a pair of brackets and put the other parts in. So $xc^2 + 4x^2 = x^2(c + 4)$.

Altogether, we now have

$$z = 3 + a + (b + 2)x + (c + 4)x^2$$

Notice two things here - I moved all those terms around in any direction depending on what was convenient. This is completley OK in addition and subtraction. After all $1 + 2$ is the same as $2 + 1$!

I also moved the x and x^2 behind the terms in the brackets. When terms are multiplied like that the order in which they appear in their multiplication doesn't matter. Again, $1 * 2$ is the same as $2 * 1$.

Rules of exponentiation

There are some rules around exponentiation that are worth understanding, even if you need to look them up the first few times you use them.

Pro tip: needing to look up an equation or a rule is no problem and not a good indication of your maths ability. You'll want to memorise the ones you use commonly, but this will happen naturally on its own anyway. If you don't have a photographic memory for equations, that's fine!

1. When multiplying two together and they have the same base, e.g. x, we can add the indices to collect terms together. If we wanted a general version of this rule, we could say $x^a x^b = x^{a+b}$.
    - Having an equation like like $ax^2 bx^3$ is pretty clunky. Let's simplify it using this rule. So $ax^2 bx^3 = abx^5$.
2. This also goes the opposite way too. If we are dividing two terms, we can subtract the indices to collect the terms together. So generally we'd say $\frac{x^a}{x^b} = x^{a-b}$.
    - Let's simplify $\frac{x^5}{x^3} = x^{5-3} = x^2$.
3. What if we're raising one exponent to the power of another exponent? We're raising a power to a power and we need a new rule for that. In this case we would say that $(x^a)^b = x^{ab}$. We multiply the two exponents together.
    - So say we had an equation like $(z^7)^7$, we could simplify that to z^{49}.
4. There's two other rules that are useful, but they're so simple they can go together. Anything raised to the power of 1 is just itself. So $x^1 = x$. In addition, anythign raised to the power of 0 is equal to 1. That is, $x^0 = 1$.
    - So for example $z^1 = z$ and $y^0 = 1$.

If you'd like to see some of these concepts in action, this is a brief video I made: https://youtu.be/lIndhkm12_E (https://youtu.be/lIndhkm12_E)

Order of operations

_ Or as I tell my students: which bit do you have to do first?_

Operations are just something that you can do to data or numbers. It's things like addition, substraction, multiplication, division, exponentiation, logs (haven't talked about those!) and so on.

Anytime you perform an action on a number and get a new number, it's probably an operation.

The order matters - not just for your calculator, but for the language of mathematics to make sense at all. For instance, the sentence *mat the on sat cat* makes no sense. In English, it must be *the cat sat on the mat* in order to be rendered correctly. It's the same in mathematics.

So what's the order?

1. Anything inside brackets comes first. Inner pairs take priority over outer pairs.
    - Basically it works as

$$[(dothisfirst) thenthis] thentheoutside$$

2. After that is exponentiation

3. Then multiplication and division
4. Then addition and subtraction

Let's take a couple of examples:

1. $\frac{30}{22}^{1/4} - 1$
   - In this case, we do the $30/22 = 1.363636$ first, then raise it to the power of $\frac{1}{4}$ so that $1.363636^{\frac{1}{4}} = 1.080624$ and then subtract 1 for a final answer of 0.080624.
2. $\frac{50}{18}^{5} + 10 * 2^4$
   - Here, we'd do $\frac{50}{18} = 2.7777$ first, then raise it to the power of 5 so that $\frac{50}{18}^{5} = 165.3794$. We'd also do {2^4 = 16} before multiplying by 10 so that $10 * 2^4 = 160$. Only THEN would we add the two terms together for a final total of $165.3794 + 160 = 325.3794$.

If you're someone who learns well with video, here's one I made here: https://youtu.be/_P6zrZyfWLY (https://youtu.be/_P6zrZyfWLY)

Rounding

Rounding is actually really important if you ever need to work with real numbers. It's just the process of taking a number with a whole bunch of digits and reducing those to something that can be meaningfully interpreted by your reader of listener.

Imagine you're listening to a presenter and he/she/they is very carefully explaining that the mean of x is 1.256422353342356455. You stopped reading after the second digit, didn't you? You'd probably stop listening too. So rounding matters if you're ever going to be communicating real findings in real numbers to real people. Rounding abbreviates a number to a more digestible form

The difference between 1.567882 and 1.6 is often greatest not in the significance of the digits, but in how it's perceived. Humans in general find it really hard to keep a whole bunch of numbers in their head, so rounding can really help you communicate your work sensibly to a broader audience.

If you'd like a video version of this topic before reading on, here is one https://youtu.be/qR5P8HEXTrY (https://youtu.be/qR5P8HEXTrY).

The basic rules of rounding are quite simple:

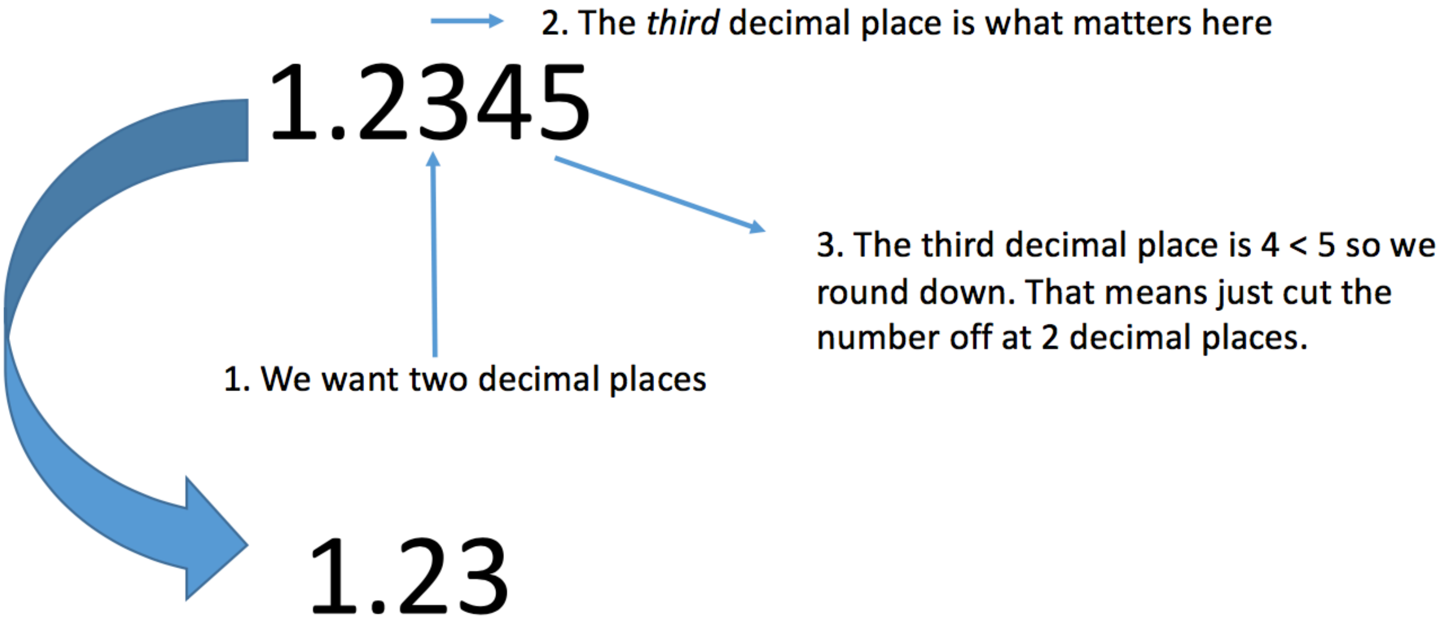
1. If the last digit we care about is 5 or greater, we round up.
2. If it is less than five, we round down.

Sometimes there is a specification to round up or down, no matter what the last digit is, in that case we do as required.

Let's look at some examples.

1. Round 1.2345 to two decimal places.
   - We want two decimal places in this case
   - So it's the third decimal place: 1.2 3 **4** 5 that matters. We decide whether or not to round the second decimal place up or down based on the third.
   - The third decimal place is $4 < 5$ so we are going to round *down*.

- Rounding down simply means ‘leave the second decimal place as it is’. We will treat the third decimal place and all that come after it as though they are zero.
- So our answer is 1.23.



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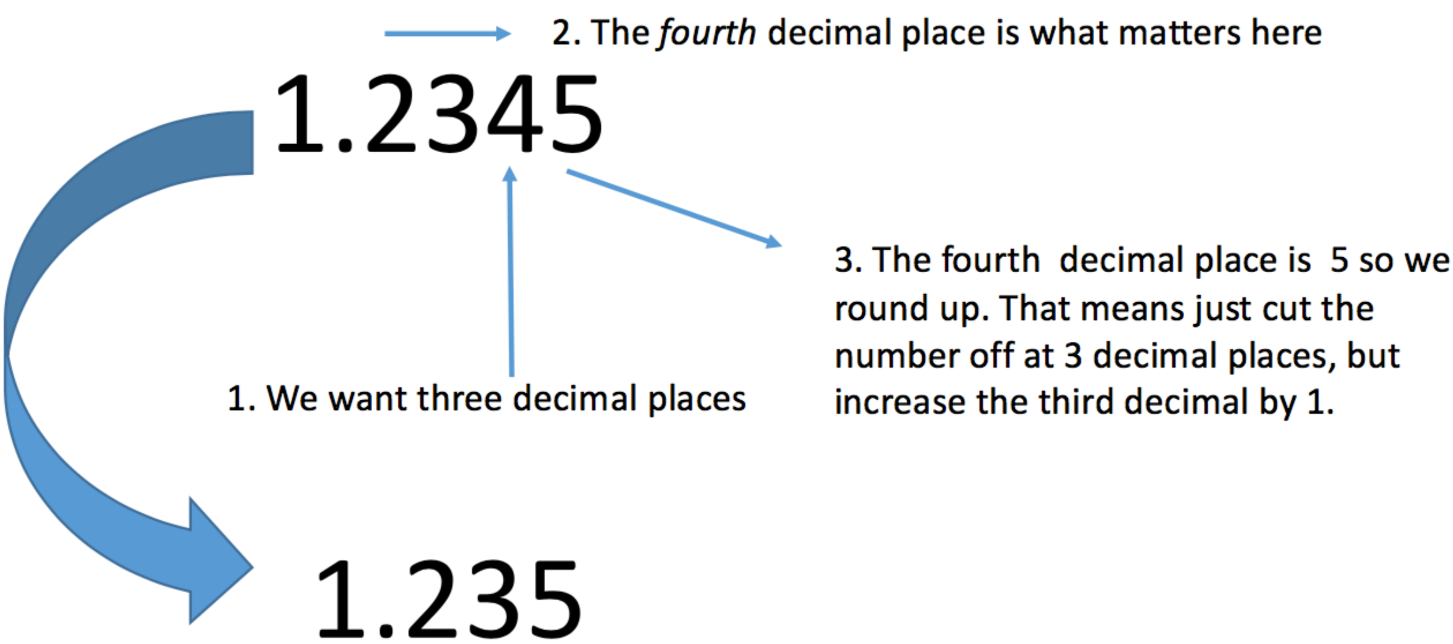
1. Round 1.2345 to three decimal places.
    - We want three decimal places in this case
    - So it’s the fourth decimal place: 1.2 34 **5** that matters. We decide whether or not to round the third decimal place up or down based on the fourth
    - The fourth decimal place is $5 = 5$ so we are going to round up
    - Rounding up here simply means ‘add one to the third decimal place’. The 4 becomes 5. We will treat the third decimal place as though it is higher by one, and all others after it as zero.



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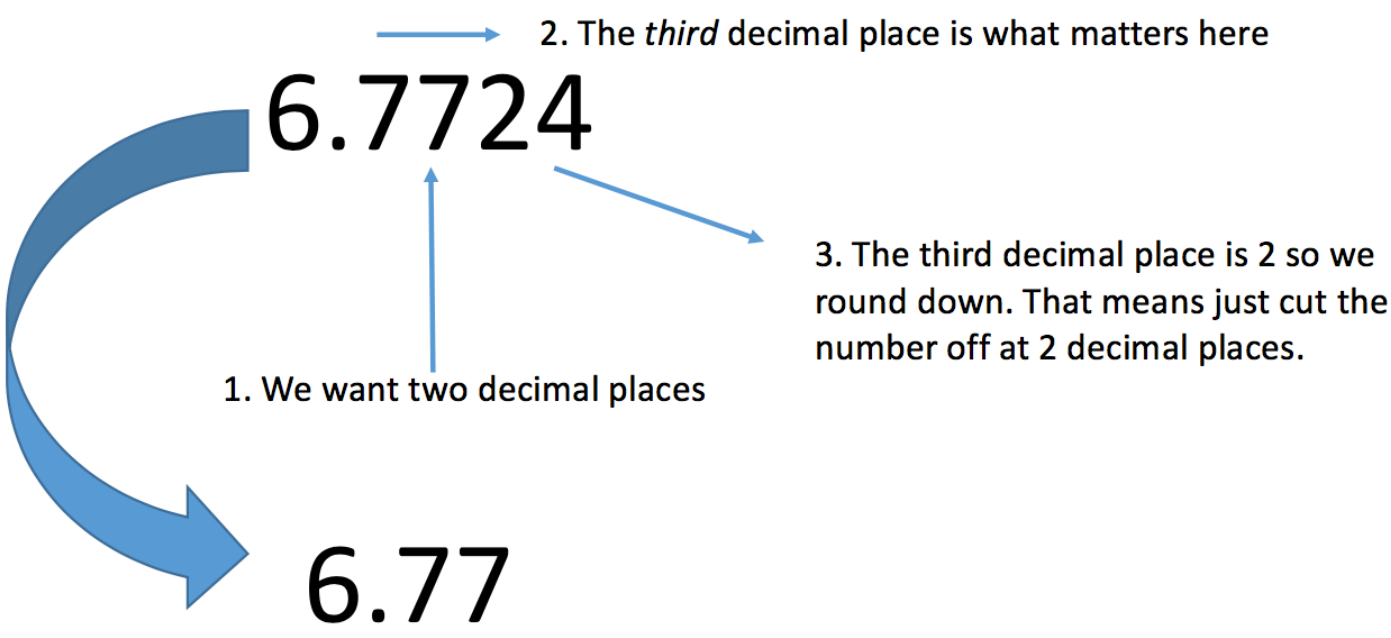
1. Round 6.7724 to two decimal places.
    - We want two decimal places in this case
    - So it’s the three decimal place: 6.77 **2** 4 that matters. We decide whether or not to round the second decimal place up or down based on the third
    - The third decimal place is $2 < 5$ so we are going to round down
    - We will treat the second decimal place as is, and pretend that all other decimals after are zero.
    - So our answer is 6.77.



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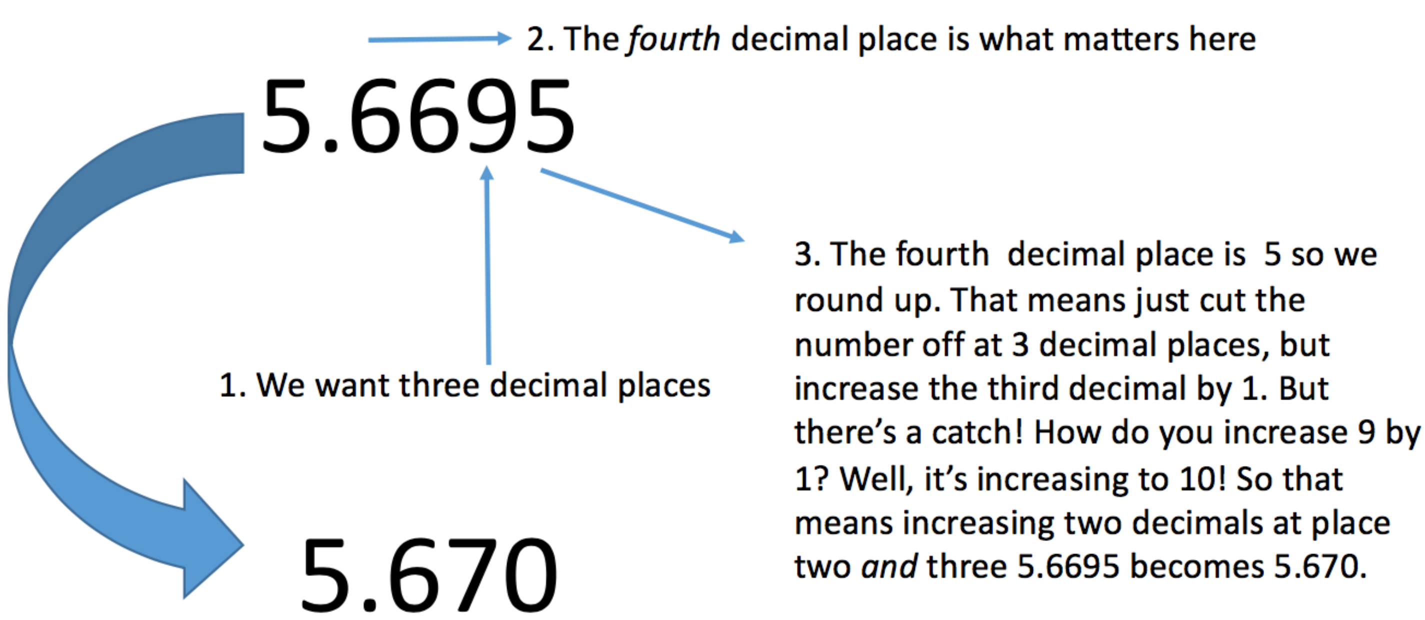
1. Round 5.6695 to three decimal places.
    - We want three decimal places in this case
    - So it's the fourth decimal place: 6.669 **5** that matters. We decide whether or not to round the third decimal place up or down based on the fourth
    - The third decimal place is $5 = 5$ so we are going to round up
    - The fourth decimal place is 5 so we round up. That means just cut the number off at 3 decimal places, but increase the third decimal by 1. But there's a catch! How do you increase 9 by 1? Well, it's increasing to 10! So that means increasing two decimals at place two and three 5.6695 becomes 5.670.
    - So our answer is 5.670.



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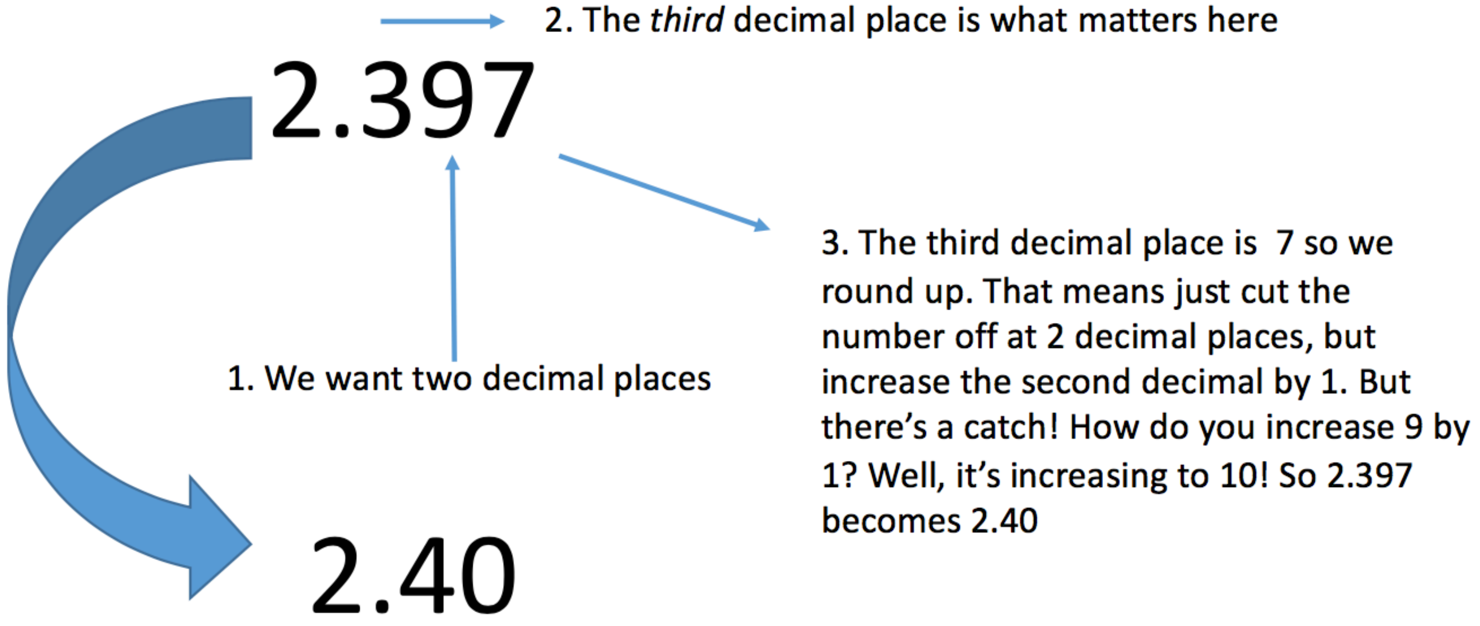
1. Round 2.397 to two decimal places.
    - We want two decimal places in this case
    - So it's the third decimal place: 2.39 **7** that matters. We decide whether or not to round the second decimal place up or down based on the third
    - The third decimal place is $7 > 5$ so we are going to round up
    - That means just cut the number off at 2 decimal places, but increase the second decimal by 1. But there's a catch! How do you increase 9 by 1? Well, it's increasing to 10!

- So 2.397 becomes 2.40.



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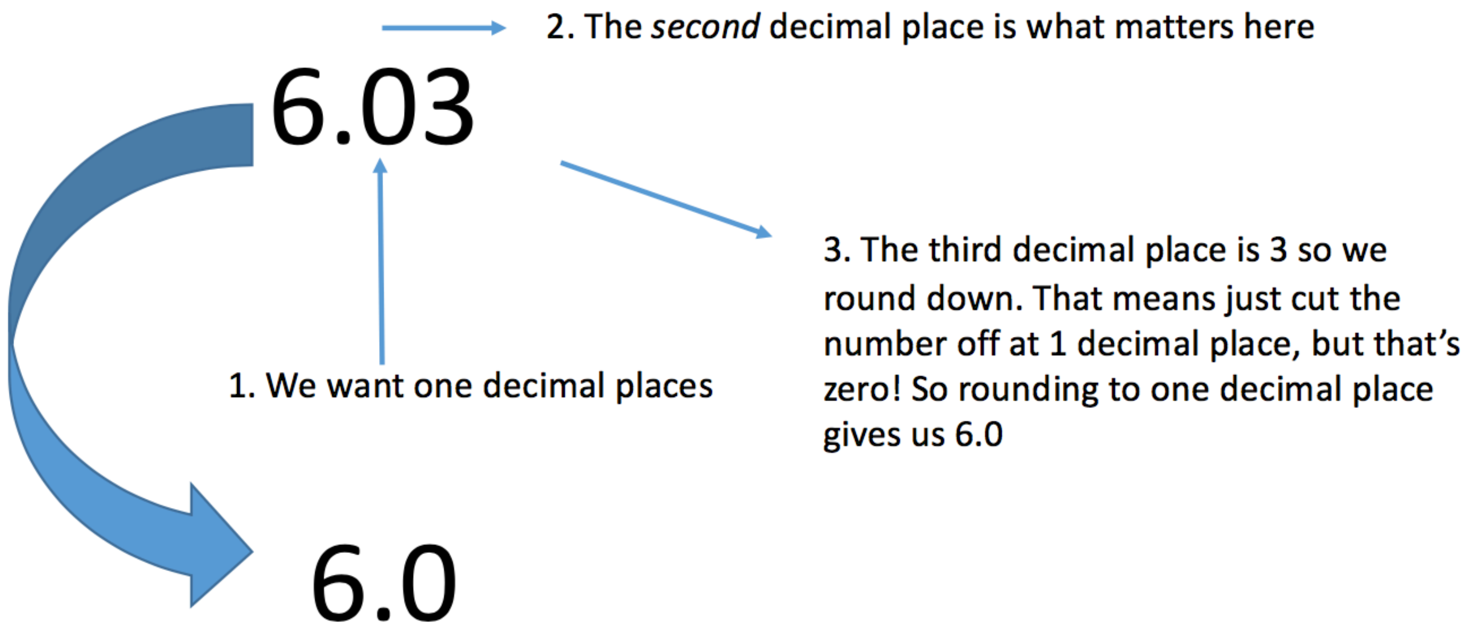
1. Round 6.03 to one decimal place.
    - We want one decimal place in this case
    - So it's the second decimal place: 6.0 **3** that matters. We decide whether or not to round the first decimal place up or down based on the second
    - The third decimal place is $3 < 5$ so we are going to round down
    - That means just cut the number off at 1 decimal place, but that's zero!
    - So rounding to one decimal place gives us 6.0

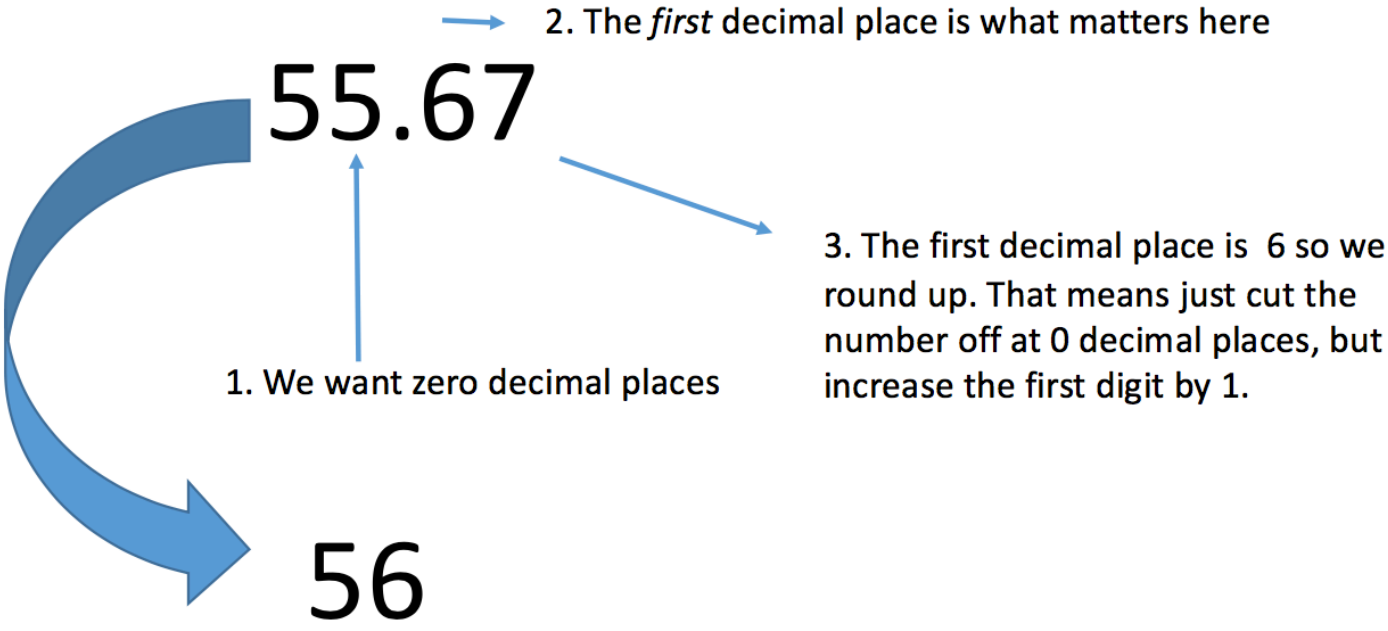


Rounding to the nearest dollar

This is just another kind of rounding, but the nearest dollar means no cents – no decimal places at all. This is the same thing as rounding to the nearest whole number.

1. Let's round $55.67 to the nearest dollar.
    - We want zero decimal places
    - The first decimal place is what matters here.
    - The first decimal place is 6 so we round up. That means just cut the number off at 0 decimal places, but increase the first digit by 1.

- The answer is $56



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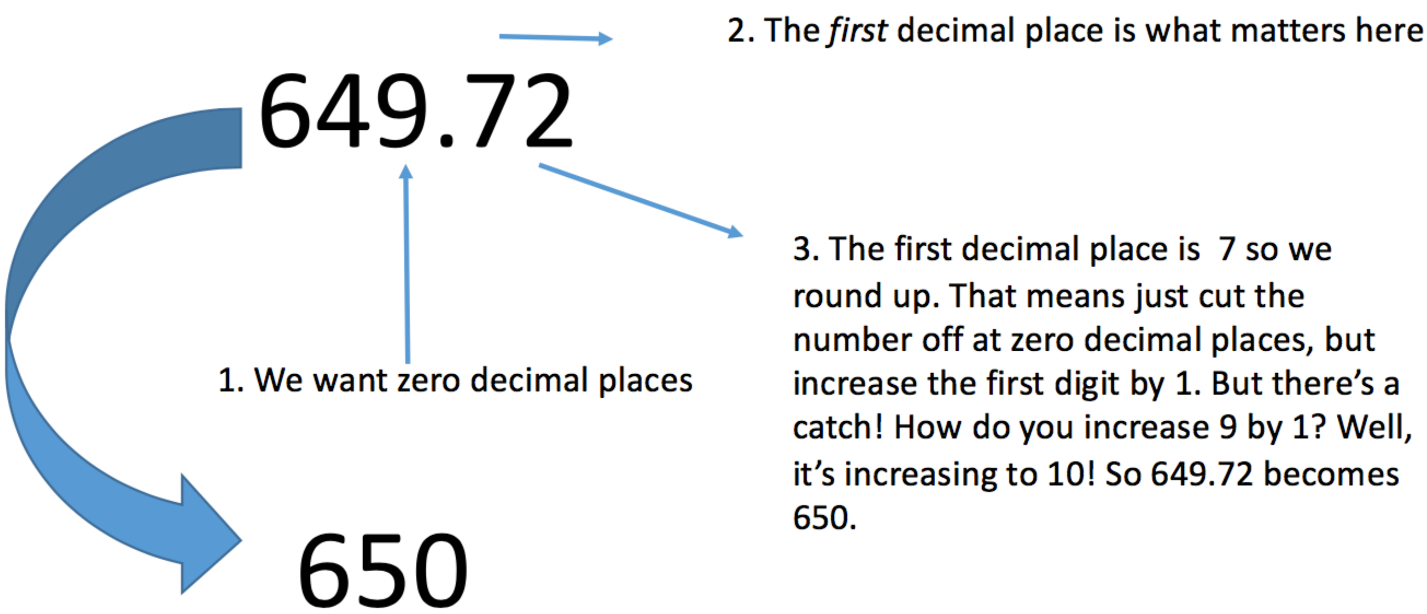
1. Let's round $649.72 to the nearest dollar
    - We want zero decimal places
    - The first decimal place is what matters here
    - The first decimal place is 7 so we round up. That means just cut the number off at zero decimal places, but increase the first digit by 1. But there's a catch! How do you increase 9 by 1? Well, it's increasing to 10!
    - So 649.72 becomes 650.



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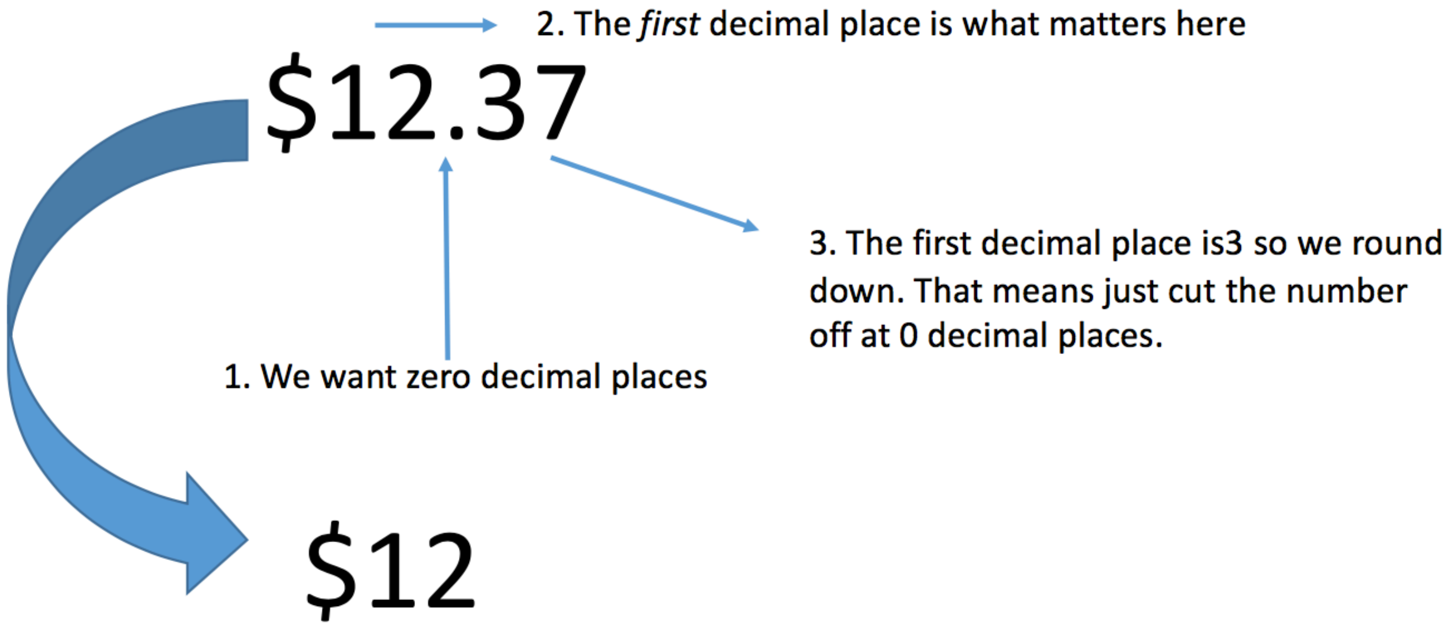
1. Let's round $12.37 to the nearest dollar
    - We want zero decimal places
    - The first decimal place is what matters here +nThe first decimal place is3 so we round down. That means just cut the number off at 0 decimal places.



And that's it. For the moment, anyway.

So that's the end of this very short introduction to algebra. Hopefully it woke up some long forgotten brain cells, or made a few things clearer on the page.

Maybe though you're in need of something more in depth, or you're interested in pursuing your maths skills further. This is *great*!

There are some good resources out there, but the one I most particularly recommend is this one:

- Eddie Woo is the head teacher of mathematics at a Sydney high school. He has a great philosophy of engaging 'maths is for everyone' teaching. He has an extensive catalogue of YouTube videos as well as other resources. If you're interested, I encourage you to check his work out: https://misterwootube.com (https://misterwootube.com)